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arithmetical values remain yet to be tested. We may thus regard the *qualitative* analysis as having been effected for these formulæ regarded as series; their *quantitative* analysis remains yet to be performed. The terminal term R is a function of an absolutely unknowable value of x, which can only be eliminated by showing that R vanishes when n is infinite. The examples given seem sufficient to illustrate the class of functions to which the section is applied, and we leave its forms for the present in order to pass now to the consideration of those more important functions which yield a difference-product as a body-determinant.

To be continued.

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SOLUTIONS OF EXERCISES.

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Discuss the case of two forces acting together upon a particle, one force being directed to the centre, and the other directed to one focus of an ellipse, under the influence of which forces the particle freely describes the ellipse.

[*Yale Prize Problem.*]

SOLUTION.

1. Prof. Curtis, in the Messenger of Mathematics, 1880, proved that, if a particle describe a curve freely under the combined action of the forces F, F', etc., acting along r, r', etc. toward fixed centres, the equation

$$\gamma\varphi d\left[\frac{F}{\varphi}\right] + \gamma'\varphi' d\left[\frac{F'}{\varphi'}\right] + \ldots = 0\,, \quad \text{or} \quad \Sigma\gamma\varphi d\left[\frac{F}{\varphi}\right] = 0$$

must be satisfied at every point of the curve, φ, φ', etc. denoting the forces respectively co-directional with F, F', etc., under which *singly* the given curve would be described; and γ, γ', etc. denoting the corresponding semichords of the circle of curvature at the point.

The proof is as follows: Putting the origin at one of the centres of force, the others being (h, k), (l, m), etc., we have

$$\frac{d^2x}{dt^2} = -F\frac{x}{r} - F'\frac{x-h}{r'} - \ldots, \quad \frac{d^2y}{dt^2} = -F\frac{y}{r} - F'\frac{y-k}{r'} - \ldots.$$

Multiplying the first equation by 2 dx, the second by 2 dy, and integrating,

$$\left[\frac{dx}{dt}\right]^2+\left[\frac{dy}{dt}\right]^2=v^2=-2\,(\int Fdr+c)-2\,(\int F'dr'+c')-\dots$$

$$=-2\,\Sigma\,(\int Fdr+c).$$

Also, since the centrifugal acceleration, v^2/ρ, at any point in the orbit, must be equal and opposite to the sum of the components of the central accelerations taken in the normal direction (ρ being the radius of curvature, and p the perpendicular from the centre of force on the tangent at any point),

$$\frac{v^2}{\rho}=F\frac{p}{r}+F'\frac{p'}{r'}+\dots,\qquad v^2=F\frac{p}{r}\rho+F'\frac{p'}{r'}\rho+\dots=F\gamma+F'\gamma'+\dots.$$

(Williamson's Diff. Calc., Art. 235).

We have, therefore,

$$v^2=\Sigma F\gamma=-2\Sigma\,(\int Fdr+c).$$

Differentiating, $$\Sigma\,(Fd\gamma+\gamma dF)=-2\Sigma\,Fdr\,,$$

or $$\Sigma\{F(d\gamma+2\,dr)+\gamma\,dF\}=0\,. \tag*{1)}$$

In particular, $$\varphi\,(d\gamma+2dr)+\gamma d\varphi=0\,,$$

$$d\gamma+2dr=-\frac{\gamma}{\varphi}\,d\varphi\,.$$

Substituting in (1),

$$\Sigma\gamma\left[dF-\frac{Fd\varphi}{\varphi}\right]=0\,,\quad\text{or}\quad\Sigma\gamma\varphi d\left[\frac{F}{\varphi}\right]=0\,. \tag{2}$$

2. We will derive another equation *free from* γ.

$$\Sigma F\gamma=-2\Sigma\,(\int Fdr+c)\,. \tag{3}$$

In particular $$\varphi\gamma=-2(\int\varphi dr+s).$$

Substituting for γ in (3),

$$\Sigma\frac{F}{\varphi}\,(\int\varphi dr+s)=\Sigma\,(\int Fdr+c)\,.$$

Differentiating,

$$\Sigma d\left[\frac{F}{\varphi}\right](\int\varphi dr+s)+\Sigma\frac{F}{\varphi}\,\varphi dr=\Sigma\,Fdr\,,\qquad\Sigma d\left[\frac{F}{\varphi}\right](\int\varphi dr+s)=0\,. \tag{4}$$

This equation might be derived directly from (2); for we have

$$\gamma_1\varphi_1 = v_1^2 = -2\left(\int\varphi_1 dr_1 + s_1\right).$$

Substituting for $\gamma\varphi$ in (2) we obtain (4).

The constant s in (4) must be determined by the particular circumstances of the problem.

Equations (2) and (4) are evidently satisfied if F is any multiple of φ. We have, therefore, the particular solution

$$F = l\varphi\,, \qquad F' = m\varphi', \qquad F'' = n\varphi'', \qquad \ldots .$$

That is, if a particle will freely describe a given curve under the separate action of any one of several forces, it will freely describe the same curve when the forces act together; or when they act together, the intensity of each being multiplied by any arbitrary number, provided it is given a proper initial velocity.

3. For the case of a particle describing an ellipse under the action of two forces, one directed to a focus and the other to the centre, (4) becomes

$$\frac{d}{dr}\left[\frac{F}{\varphi}\right] dr\left(\int\varphi dr + s\right) + \frac{d}{dr'}\left[\frac{F'}{\varphi'}\right] dr'\left(\int\varphi' dr' + s'\right) = 0\,, \tag{5}$$

where $\varphi\ = \mu r^{-2} =$ force toward left-hand focus,
$\varphi' = \mu' r' =$ " " centre.

To determine s and s' we suppose the particle to be acted on separately by φ and φ' and to be at the right hand extremity of the major axis. Evidently $\gamma = \gamma' = \rho = \dfrac{b^2}{a}$.

In the first case, if v_0 be the velocity,

$$v_0^2 = \varphi\gamma = -2\int\varphi dr - 2s = \frac{2\mu}{r} - 2s\,,$$

$$\frac{\mu}{(a+c)^2}\frac{b^2}{a} = \frac{2\mu}{a+c} - 2s\,, \quad 2s = \frac{\mu}{a}\,.$$

In the second case, if v_1 be the velocity,

$$v_1^2 = \varphi'\gamma' = -2\int\varphi' dr' - 2s' = -\mu' r'^2 - 2s'\,,$$

$$\mu' a\frac{b^2}{a} = -\mu' a^2 - 2s'\,,$$

$$2s' = -\mu'(a^2 + b^2)\,.$$

To obtain a relation between r and r', we have, putting the origin at the centre,

$${}^{2} = x^2 + 2cx + c^2 + y^2 ,$$

$$r'^2 = x^2 + y^2 ,$$

$$r^2 - r'^2 = 2cx + c^2 ,$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 ,$$

$$c^2 = a^2 - b^2 .$$

From these equations we obtain

$$r'^2 - (r - a)^2 = b^2 , \tag{6}$$

$$r'dr' = (r - a)\, dr .$$

By substituting in (5) their values for φ, φ', s, s' and dr' we obtain

$$\frac{1}{r^2(a - r)} \frac{d}{dr} (Fr^2) = \frac{a}{r'} \frac{d}{dr'} \left[\frac{F'}{r'}\right] . \tag{7}$$

This is satisfied by the equations

$$\frac{1}{r^2(a - r)} \frac{d}{dr} (Fr^2) = f_1 \left[(r - a)^2\right] + f_2 \left[(r - a)^2 + b^2\right] ,$$

$$\frac{a}{r'} \frac{d}{dr'} \left[\frac{F'}{r'}\right] = f_1 (r'^2 - b^2) + f_2 (r'^2),$$

where f_1, f_2 are any functions. If particular forms are given to these functions, corresponding forms of F and F' are obtained by clearing the equations of fractions, and integrating.

4. We will now solve for F and F' using (2).

To find a relation between γ and γ', putting the origin at the left focus, we have (Williamson's Diff. Calc. Art. 172)

$$\frac{(x - c)^2}{a^2} + \frac{y^2}{b^2} = 1 , \quad x^2 + y^2 = r^2 ,$$

$$p^2 \left[\frac{(x - c)^2}{a^4} + \frac{y^2}{b^4}\right] = \left[\frac{x(x - c)}{a^2} + \frac{y^2}{b^2}\right]^2 ;$$

whence

$$p^2 = \frac{b^2 r}{2a - r} ,$$

$$\gamma = p \frac{dr}{dp} = \frac{r(2a - r)}{a} .$$

Now, putting the origin at the centre, we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,,\quad x^2+y^2=r'^2\,,\quad \frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{p'^2}\,;$$

whence
$$a^2+b^2-r'^2=\frac{a^2b^2}{p'^2}\,,$$

$$\gamma'=p'\frac{dr'}{dp'}=\frac{a^2+b^2-r'^2}{r'}=\frac{r\,(2a-r)}{r'}\,,\ \text{by (6)},$$

$$\frac{\gamma'}{\gamma}=\frac{a}{r'}\,.$$

We have now

$$\gamma\varphi\frac{d}{dr}\left[\frac{F}{\varphi}\right]dr+\gamma'\varphi'\frac{d}{dr'}\left[\frac{F'}{\varphi'}\right]dr'=0$$

$$\gamma'=\frac{a\gamma}{r'}\,,\quad \varphi=\frac{\mu}{r^2}\,,\quad \varphi'=\mu'r'\,,\quad r'dr'=(r-a)dr\,.$$

Making the proper substitutions we obtain

$$\frac{1}{r^2(a-r)}\frac{d}{dr}\,(Fr^2)=\frac{a}{r'}\frac{d}{dr'}\left[\frac{F'}{r'}\right],$$

the same as (7). [*Joseph Bowden, Jr.*]

EXERCISES.

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PROVE that, if $0<\alpha<\beta$,

$$\int_\alpha^\beta \log\frac{\beta-x}{x-\alpha}\,\frac{dx}{x}=\frac{1}{2}\left[\log\frac{\beta}{\alpha}\right]\,.$$

[*Frank Morley.*]

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INTEGRATE the differential equation

$$dy=\text{arc sin}\,(x^2)\,dx.$$

[*Artemas Martin.*]